Fractals: An Introduction

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Fractals are intricate mathematical objects, many of which bear a startling resemblance to natural objects and phenomena. They have been described as "a rough or fragmented geometric shape that can be split into parts, each of which is (at least approximately) a reduced-size copy of the whole" [2]. Their characteristics include a highly irregular form, self-similarity, detailed structure across all scales and a fractal dimension which is generally larger than the topological dimension. Despite these complex attributes, fractals are often constructed using a deceptively simple iterative rule.

Fractals are commonly associated with Chaos Theory, the study of dynamical systems which exhibit extreme sensitivity to initial conditions: small differences in the initial state result in profoundly different outcomes. Like fractals, this complex behaviour arises from a deterministic rule and does not involve any element of randomness. Whereas many dynamical systems evolve towards either a fixed point or a limit cycle, chaotic systems are drawn towards strange attractors, which have fractal structure.

One of the defining characteristics of fractals is self-similarity: their appearance remains similar at a variety of scales. An illustrative example is a coastline: if one were to imagine descending towards an island from a great height, progressively more detailed structure would be revealed as one got closer and closer, yet the qualitative appearance of the coastline would remain unchanged.

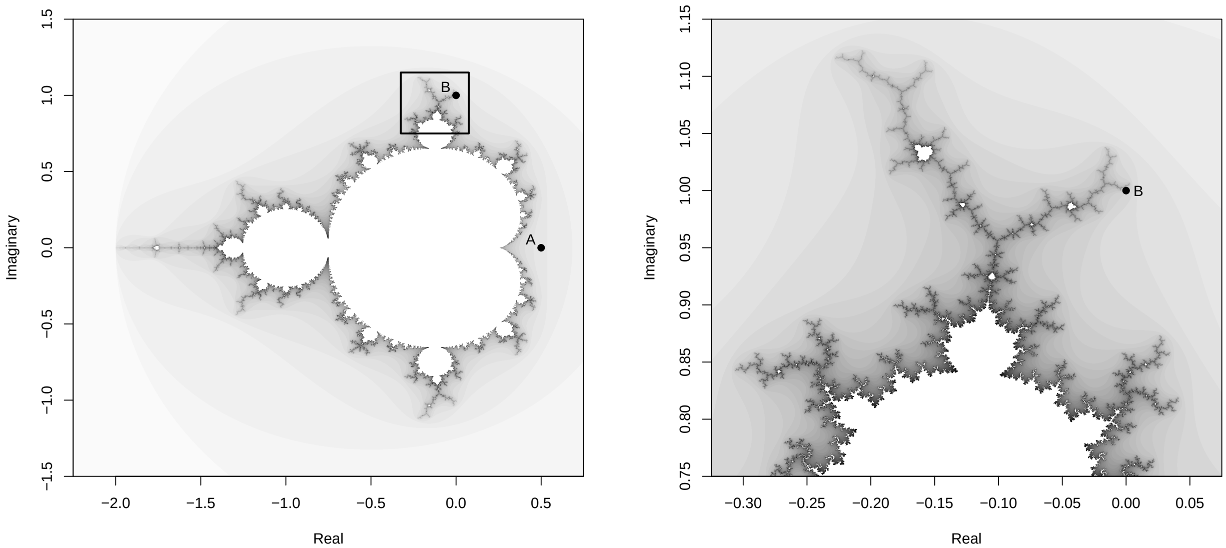
The Cantor Set and Koch Snowflake

The Cantor Set is one of the earliest examples of a fractal. It is constructed iteratively from a line segment of unit length. In the first stage the middle 1/3 interval of the line is removed, leaving two segments, each of length 1/3. In the next stage, the same process is applied to each of these segments. The process is then repeated indefinitely. The Cantor Set contains all of the points which were not removed. The first six steps in this process are illustrated in Figure 1, where the self-similar structure is readily apparent.

The Cantor Set has some astonishing attributes: it does not include any intervals of non-zero length (the total length of the intervals removed is equal to the length of the original interval) but it contains an infinite number of points (in fact, it contains as many points as the original interval!). These features arise because the intervals being removed in each iteration are "open": they do not include their end points.

A related fractal, the Koch Snowflake, is constructed from a comparably simple rule: starting with an equilateral triangle, one modifies each side as follows: (i) divide the side into three equal segments; (ii) remove the

Figure 1: The first six stages in the construction of the Cantor Set.



(a) (b)

Figure 2: The Mandelbrot Set, where points within the set are plotted in white and the gray scale indicates the rate of divergence for points outside the set. The image in (b) corresponds to the region indicated by the black square in (a). The points A and B are at $0.5 + 0i$ and $0 + 1i$ respectively.

middle segment and (iii) replace it with an equilateral triangle pointing outward. Repeated application of this rule results in an infinitely convoluted fractal curve which resembles a snowflake.

The Mandelbrot Set

The Mandelbrot Set has become an iconic symbol for fractals. Figure 2a is an image of the full Mandelbrot Set, which represents a detailed collection of points in the plane of complex numbers. Whether or not a point $c \in \mathbb{C}$ belongs to the set is determined by the complex series defined by the recurrence relation

$$z_{n+1} = z_n^2 + c$$

where $z_0 = 0$. If $|z_n|$ remains bounded as $n \to \infty$, then c lies within the set. For example, if $c = 0.5$ (point A in Figure 2) then the series is 0, 0.5, 0.75, 1.0625, $\ldots$ which diverges rapidly. In contrast, $c = i$ (point B in Figure 2) results in the series $i - 1$, $-i$, $i - 1$, $\ldots$ which is clearly bounded. The nearby point at $0.000000001 + i$ results in a diverging sequence and thus lies outside the set, illustrating the sensitivity of the process to initial conditions.

As illustrated in Figure 2b, the Mandelbrot Set displays progressively finer detail at higher levels of magnification, while the boundary of the set includes innumerable smaller copies resembling the set as a whole. This is another example of the fractal property of self-similarity.

Fractal Dimension

Fractals have a Hausdorff or fractal dimension which is not necessarily integral and generally exceeds their topological dimension. The fractal dimension reflects how effectively a fractal occupies space and is intimately linked to its scaling characteristics. It can be thought of as an index characterising the relationship between change in detail and change in scale. The Cantor Set has fractal dimension of $\log(2)/\log(3) \approx 0.6309$, which lies between that of a point (zero dimensional) and a line (one dimensional). The Koch Snowflake, which has a topological dimension of 1, consists of an infinite number of infinitesimal line segments: it fills space more efficiently than a simple line, yet it is not complex enough to be two dimensional. It has a fractal dimension of $\log(4)/\log(3) \approx 1.2619$. Similarly, a coastline, which reveals progressively more complexity as the length scale decreases, is more complicated than a simple curve. The coastline of Great Britain has fractal dimension of 1.25, while that of Norway has dimension 1.52. This suggests that the Norwegian coastline is more convoluted than that of Great Britain and reference to a map confirms it! Somewhat counter-intuitively, the Mandelbrot Set has a fractal dimension of 2, filling the plane as effectively as a two dimensional object. The human cerebellum has a fractal dimension of 2.57[1], quite similar to that of 2.66 dimensional broccoli.

Fractals in Nature

Most images of fractals are numerically generated. However, there are numerous observations of phenomena in nature which exhibit fractal properties[2]. For example, lightning, coastlines, clouds, snow flakes, some vegetables and heart rhythms all have fractal characteristics. The analogy between nature and fractals has been extremely fruitful. Fractals can be used to gain a deeper understanding of nature. They can also be applied in computer simulations and are routinely used to generate realistic digital landscapes and cloud formations.

Applications of Fractals

Fractals have inspired a myriad of applications. To list only a few: (i) they have been applied to the algorithmic creation of music; (ii) some lossy compression techniques for digital images are based on fractal decomposition; and (iii) novel antennas for transmitting or receiving radio waves have been designed using the space filling characteristics of fractals to maximise antenna length within a confined volume.

Conclusion

In addition to being curious mathematical oddities, fractals have applications across a range of topics in science and engineering. With continued improvements in computational power, we are likely to uncover even more uses for these attractive and fascinating objects.

References

[1] Jing Z Liu, Lu D Zhang, and Guang H Yue. Fractal Dimension in Human Cerebellum Measured by Magnetic Resonance Imaging. *Biophysical Journal*, 85(6):4041–4046, December 2003.

[2] Benoit B Mandelbrot. *The Fractal Geometry of Nature*. W. H. Freeman and Company, 1983.